f^{-1} series generated by using the branching process model

1989 J. Phys. A: Math. Gen. 22 L931

(http://iopscience.iop.org/0305-4470/22/19/003)

J. Phys. A: Math. Gen. 22 (1989) L931–L938. Printed in the UK

LETTER TO THE EDITOR

f^{-1} series generated by using the branching process model

T Kobayashi

Department of Physics, Shiga University of Medical Science, Ohtsu, Shiga 520-21, Japan

Received 18 January 1989

Abstract. The branching process model was applied to generate the time series for a number of particles existing in a system and for counts recorded by a detector placed in the system. The power spectral density of the time series for the particle number is characterised by a f^{-2} distribution, but the series made by time intervals between the successive counts has a f^{-1} spectrum in a wide range of frequencies.

Since the observation of the f^{-1} power spectral density (PSD) of shot noise by Johnson in 1925 [1], many works have been published to elucidate the phenomenon theoretically [2–6], where f represents the frequency of the phenomenon. Most of them, however, are insufficient to simulate the phenomenon with a f^{-1} PSD covering a wide range of frequencies or are too sophisticated. The aim of the present letter is to give a model for generating a series in which PSD is characterised by a f^{-1} distribution for a wide range of frequencies.

When a PSD of events $x(t)$ behaves like $f^{-\gamma}$ ($\gamma \geqslant 0$), a large γ results in a long-term correlation between the events because the intensities of fluctuations with low frequencies are relatively large compared with those with high frequencies. In the case that $0 \leqslant \gamma < 1$, the correlation function is given by

$$\langle x(t)x(t+\tau)\rangle = \begin{cases} \tau^{\gamma-1} & 0<\gamma<1 \\ \delta(\tau) & \gamma=0 \end{cases} \tag{1}$$

which is independent of t, and the phenomenon is stationary. When $\gamma > 1$, the correlation function depends on t and the non-stationary phenomenon occurs. Therefore, phenomena with a f^{-1} PSD are intermediate between stationary and non-stationary phenomena.

The basic idea of the present work is to generate the series with a PSD behaving like f^{-1} by using the branching process model. Here an event may have correlation with other events through the branching processes. The branching process model was first applied in 1874 to discuss the statistics of family lines [7]. This problem has been developed by Kendall [8] and Bellman and Harris [9]. Many other applications of the branching process model have been made to the discussion of the fluctuation of the neutron number in a nuclear reactor [10, 11] and that of electron number in solids [12], to the discussion of the particle spectra obtained by a high-energy collider [13, 14], and so on. The details of the theory and variety of applications are described in the standard texts [15].

For a system in which a particle may be subjected to capture and branching reactions, the probability $P_k(n, t)$ that n particles are found in the system at time $t > 0$ after we had k particles at $t = 0$ in the presence of random particle immigration with the rate S is given in [16] as

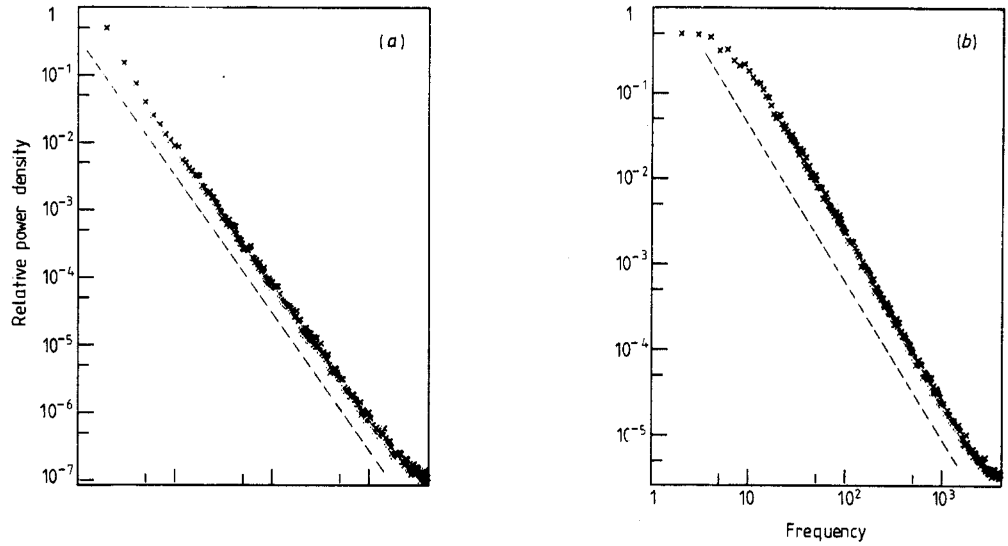
$$P_k(n, t) = \sum_{i=0}^{n} K_k^{(n-i)} Q_0^{(i)} \tag{2}$$

where $K_k^{(n-i)}$ and $Q_0^{(i)}$ are the contributions of the k particles in the system at $t = 0$ and of the particles immigrating into the system during the time interval $(0, t)$, respectively. They are given by equations (47), (48), (95) and (97) of [16] in the case of binary branching process.

The Monte Carlo method using the probability described by (2) was applied to generate the time series for the number of existing particles in the system. The generation of the time series was started from the initial particle number $N_0 = 10$. In order to avoid the possibility of the particle number increasing to infinity or dying out, the random immigration rate was chosen to be $S = \alpha N_0$ when $\mu < 1$ (subcritical case), considering the mean number of particles at $t \to \infty$ is S/α, and $S = 0$ when $\mu = 1$ (critical case). Here $\alpha = \lambda_c - \lambda_m$ and $\mu = \lambda_m/\lambda_c$ where λ_c and λ_m are the capture and branching rates, respectively, of a particle. If $S > 0$ in a critical system, the mean particle number will increase with time and diverge eventually to infinity. Three examples of the PSD of the times series generated by (2) are shown in figure 1. The PSD behaves like f^{-2} when $\mu = 1$. In the case that $\mu < 1$, the PSD converges to a finite value in a low-frequency range although it retains the f^{-2} form in the high-frequency range. The frequency range with the finite PSD value becomes wider with decreasing μ. The f^{-2} behaviour in the high-frequency range reflects the fact that the particle number at a time t_1 may have a determinant influence on the particle number at a following time t_2 even when $\mu = 0$, i.e. the correlation between the particle numbers at t_1 and t_2 is strong for a short time interval $t_2 - t_1$ and weak for a long time interval. The frequency range of the f^{-2} behaviour due to the strong correlation increases with μ and eventually covers the whole frequency range at $\mu = 1$.

It was noticed in the above analysis that the PSD of time series described by the number of particles existing in a system behaves like f^{-2} due to strong correlations between the particle numbers. However, we may expect a $f^{-\gamma}$ behaviour $(0 < \gamma < 2)$ for the series formed by time intervals between successive events as shown in figure 2 where detections of a particle are considered. A detection may correlate with another detection through branching paths as the detections a, b and c in figure 2. The detection d has no correlation with a, b and c, because it appears in a branching chain originating from a particle immigration different from that for the detections a, b and c. The length of the path between the detections has stochastical correlation with the physical time interval. For example, the time interval between a and b is approximately equivalent to that between b and c, but, owing to the fact that the path between b and c is longer than that between a and b, the correlation between b and c may be far weaker than that between a and b. These considerations motivate an analysis of series formed by time intervals between successive detections of a particle.

It is rather complicated even in the case of binary branching to give the general form of the probability $P_k(m, n, t)$ that m counts have been recorded by a detector of absorption type placed in the system during the time interval $(0, t)$ and n particles are found in the system at time $t > 0$ after we had k particles at $t = 0$ [16]. In the case that $m = 0$, however, the probability $P_k(0, n, t)$ for binary branching is described closely



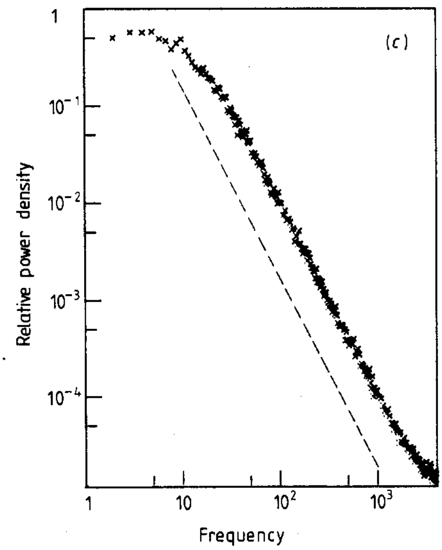


Figure 1. The PSD of the time series for existing particle number in the cases that: (a) $N_0=10$, $\mu=1$ and $\lambda_m t=0.01$; (b) $N_0=10$, $\mu=0.5$ and $\alpha t=0.02$; (c) $N_0=10$, $\mu=0$ and $\alpha t=0.01$. In each case the broken line gives the f^{-2} behaviour. The value of t is the same in (b) and (c).

by a similar form to (2) as

$$P_k(0, n, t)=\sum_{i=0}^{n} K_k^{(0,n-i)} R_0^{(0,i)} \tag{3}$$

where

$$R_0^{(0,0)}=\exp\left((\zeta-1)St+\frac{\beta-1}{\alpha} S \ln \frac{\eta-\zeta}{\eta-\zeta\,\mathrm{e}^{-\theta t}}\right) \tag{4}$$

and

$$R_0^{(0,i)}=\frac{(\beta-1)S/\alpha+i-1}{i} QR_0^{(0,i-1)} \qquad i\geqslant 1 \tag{4'}$$

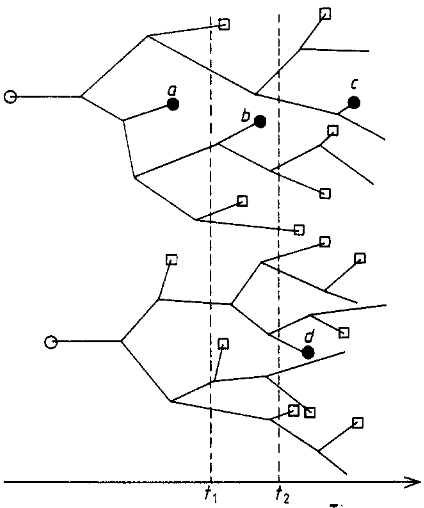


Figure 2. Chains of the branching processes. The circles represent the particles immigrating randomly, boxes represent absorption of particles and the black spots represent detection of a particle. The paths (loci) of the particles are shown by full lines. The existing particle number at times t_1 and t_2 are 8 and 13, respectively.

where

$$Q=\frac{1-e^{-\theta t}}{\eta-\zeta\, e^{-\theta t}}. \tag{5}$$

Here the parameters θ, η and ζ are defined as

$$\theta=\frac{\alpha}{\beta-1}\sqrt{(\beta-1)^2+4\beta\varepsilon} \tag{6a}$$

$$\eta=\tfrac{1}{2}[\beta+1+\sqrt{(\beta-1)^2+4\beta\varepsilon}] \tag{6b}$$

$$\zeta=\tfrac{1}{2}[\beta+1-\sqrt{(\beta-1)^2+4\beta\varepsilon}] \tag{6c}$$

in the case that $\beta>1$ (subcritical case) where

$$\beta=\frac{1}{\mu}=\frac{\lambda_c}{\lambda_m} \qquad \varepsilon=\frac{\lambda_d}{\lambda_a}. \tag{7}$$

In (7), λ_a and λ_d are the absorption and detection rates for a single particle, respectively, and $\lambda_c=\lambda_a+\lambda_d$ is the capture rate already described. In the case that $\beta=1$ (critical case)

$$\theta=2\sqrt{\varepsilon}(\lambda_m) \tag{8a}$$

$$\eta=1+\sqrt{\varepsilon} \tag{8b}$$

$$\zeta=1-\sqrt{\varepsilon}. \tag{8c}$$

The function $K_k^{(0,n-i)}$ in (3) is calculated from the following relations successively:

$$K_k^{(0,n-i)}=\sum_{l=0}^{n-i} p(0,l,t)K_{k-1}^{(0,n-i-l)} \tag{9a}$$

$$K_0^{(0,n-i)}=\delta_{0,n-i}. \tag{9b}$$

The function $p(0,l,t)$ in (9) is the probability that the detector counts no particle during the time interval $(0,t)$ and l particles are found in the system at $t>0$ when one particle has immigrated at $t=0$, and is expressed as

$$p(0, l, t)=\begin{cases}\eta\zeta Q & l=0\\ \dfrac{(\eta-\zeta)^2\,\mathrm{e}^{-\theta t}}{(\eta-\zeta\,\mathrm{e}^{-\theta t})^2} & l=1\\ Qp(0, l-1, t) & l\geqslant 2.\end{cases}\tag{10}$$

Using (2) and (3), the probability that the detector counts particles during the time interval $(0, t)$ and n particles are found in the system at $t>0$ after we had k particles at $t=0$ is given by

$$\sum_{m=1}^{\infty} P_k(m, n, t)=P_k(n, t)-P_k(0, n, t).\tag{11}$$

When the probability given by (11) is much smaller than $P_k(0, n, t)$, the probability recording more than two counts may be negligible and the following relation holds approximately:

$$P_k(1, n, t)\doteqdot P_k(n, t)-P_k(0, n, t).\tag{12}$$

Whether a particle detection has occurred or not in a very short time interval was decided successively by using the Monte Carlo method with the probabilities described by (2), (3) and (12), from which another series (count series) formed by the time intervals between successive detections was obtained. The random immigration rate S was chosen in a similar way to the time series for the existing particle number. A part of the count series is shown in figure 3 in comparison with the series for particle

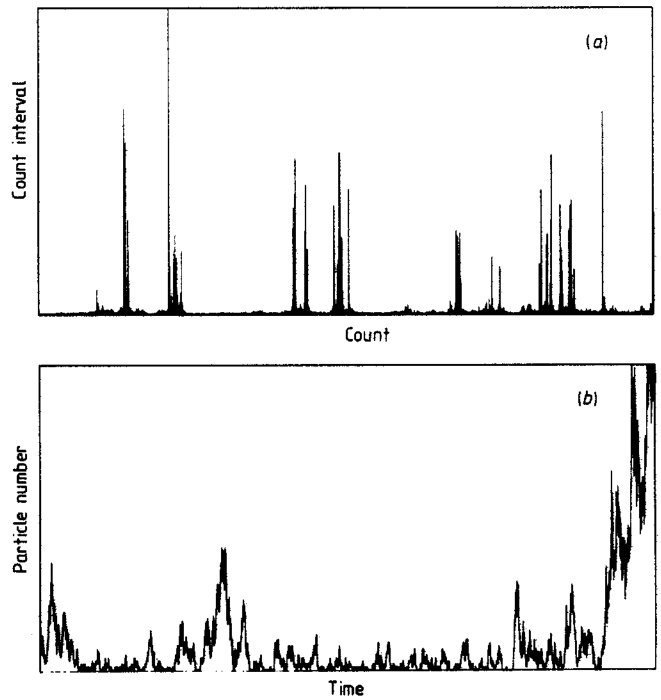
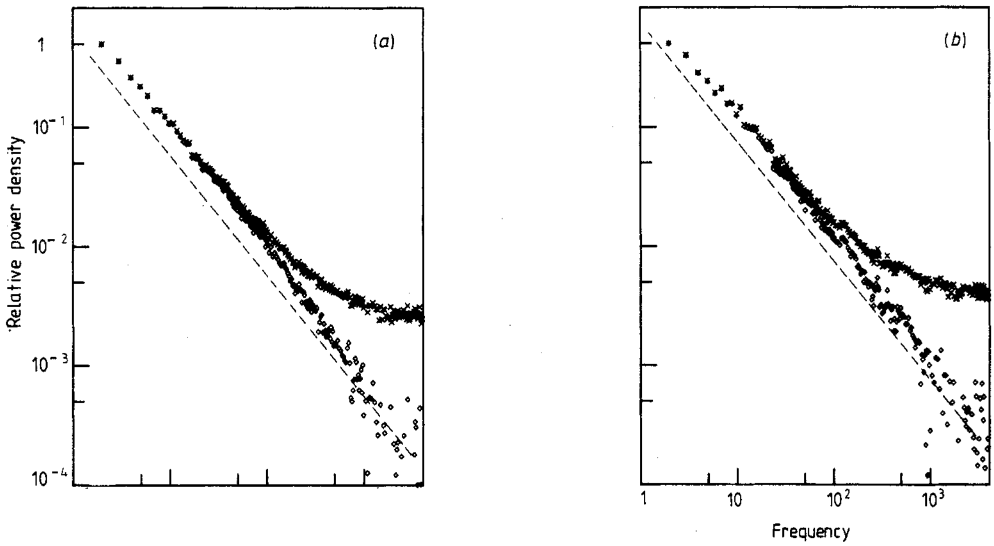


Figure 3. (*a*) Count series in the case that $N_0=10$, $\mu=1$, $\lambda_m t=0.0015$ and $\varepsilon=1$. (*b*) Time series for existing particle number in the same case as figure 1(*a*).



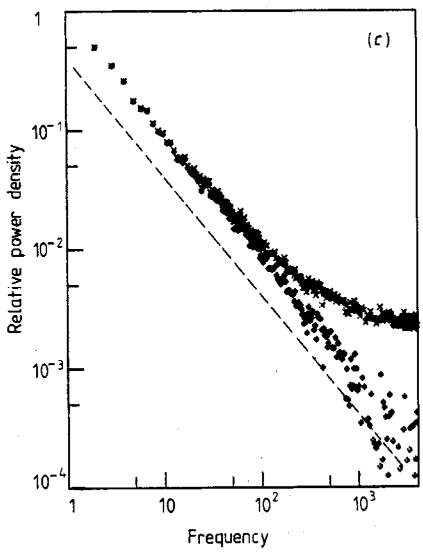


Figure 4. The PSD of the count series in the cases: (a) that $N_0 = 10$, $\mu = 1$, $\lambda_m t = 0.0015$ and $\varepsilon = 1$; (b) that $N_0 = 10$, $\mu = 1$, $\lambda_m t = 0.0002$ and $\varepsilon = 1$; (c) that $N_0 = 1$, $\mu = 1$, $\lambda_m t = 0.0015$ and $\varepsilon = 1$. The broken lines give the f^{-1} behaviour. The crosses represent the PSD calculated from the count series and the open diamonds are the PSD with the white noise subtracted.

number. The count series has a more intermittent property than the other. The PSD of the count series in the case that $\mu = 1$ and $\varepsilon = 1$ is given in figure 4(a, b). In the high-frequency range, the PSD converges to a finite value which is the white noise component of the spectrum. In the figure is also shown the PSD with the white noise component subtracted. The f^{-1} behaviour of the PSD is clearly evident in the figure over three decades of frequency. In order to examine the effect of the initial particle number N_0 on the PSD, the count series in the case that $N_0 = 1$ was generated, the PSD of which is shown in figure 4(c). No significant difference is noticed between the results shown in figure 4(a) and in figure 4(c). When $\varepsilon < 1$, however, some effect of N_0 on the PSD is noticed as shown in figure 5, where the PSD for $N_0 = 10$ is smaller in a low-frequency range than that expected from a f^{-1} distribution. When $N_0 \neq 1$, there are several independent branching chains in the system originated by the different immigrating particles as shown schematically in figure 2, which may sometimes result

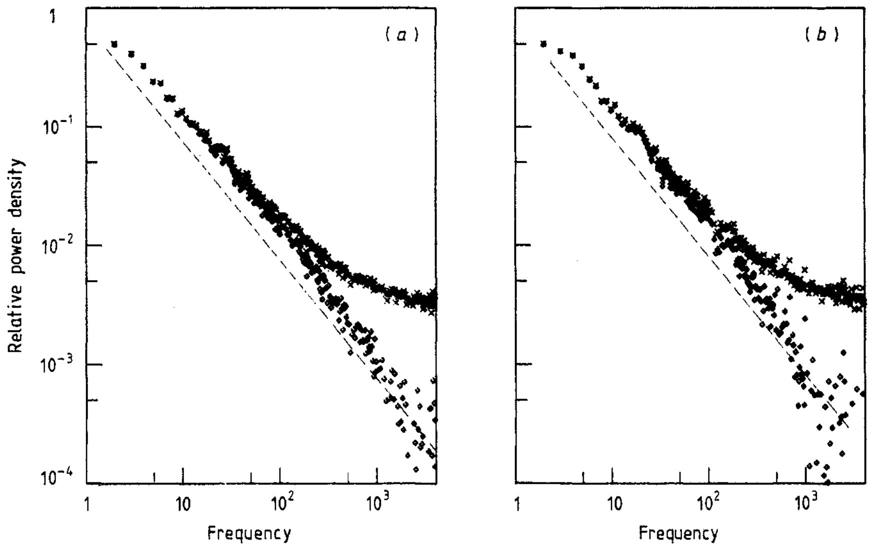


Figure 5. The PSD of the count series in the case that $\mu = 1$, $\lambda_m t = 0.003$, $\varepsilon = 0.5$ and (a) $N_0 = 1$, (b) $N_0 = 10$. The broken lines give the f^{-1} behaviour. The data points are defined as in figure 4.

in a particle detection without any correlation with other detections. While every capture event is detected in the case that $\varepsilon = 1$, only some of them, i.e. λ_d/λ_c events in all the capture events, are detected when $\varepsilon < 1$. Therefore the chance that two successive detections appear on the different branching chains increases with decreasing ε. This fact results in the low PSD in the low-frequency range in figure 5(b).

In figure 6 is shown the PSD for $\mu < 1$, which converges to a finite value in a low-frequency range due to a similar reason to that in the case of time series of particle number.

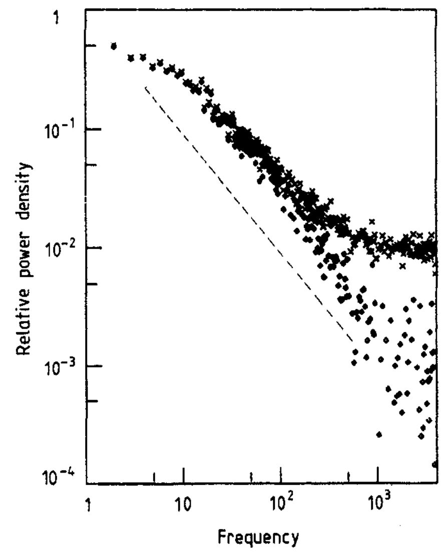


Figure 6. The PSD of the count series in the case that $N_0 = 10$, $\mu = 0.9$, $\alpha t = 0.0003$ and $\varepsilon = 1$. The broken lines give the f^{-1} behaviour. The data points are defined as in figure 4.

The series with the f^{-1} PSD is obtained in the present simulation by paying attention to the time intervals of successive detections of a particle among the many existing particles. In the present work, we consider a particle detector of absorption type placed in the system. Other types of detectors, for instance a detector of the branching events, may be available for generating series with a f^{-1} PSD. Therefore the conclusion obtained in the present work could probably be generalised to the conclusion that the f^{-1} PSD is obtained when intervals of successive events of a particular phenomenon are considered among many non-stationary phenomena with a f^{-2} PSD.

References

[1] Johnson J B 1925 *Phys. Rev.* **26** 71
[2] Pellegrini B 1980 *Phys. Rev.* B **22** 4684; 1981 *Phys. Rev.* B **24** 7071; 1982 *Phys. Rev.* B **26** 1791
[3] Rammal R, Thannous C, Breston P and Tremblay A-M S 1985 *Phys. Rev. Lett.* **54** 1718
[4] Furukawa H 1986 *Phys. Rev.* A **34** 2315
[5] Geisel T, Zacherl A and Radons G 1987 *Phys. Rev. Lett.* **59** 2503; 1988 *Z. Phys.* B **71** 117
[6] Grüneis F 1984 *Physica* **123A** 149
[7] Watson H W and Galton F 1874 *J. Anthrop. Inst.* **4** 138
[8] Kendall D G 1948 *Ann. Math. Stat.* **19** 1
[9] Bellman R and Harris T E 1952 *Ann. Math.* **55** 280
[10] Bell G I 1965 *Nucl. Sci. Eng.* **21** 390
[11] Kobayashi T 1968 *J. Nucl. Sci. Tech.* **5** 145
[12] Lin D L 1981 *J. Appl. Phys.* **52** 3575
[13] Carruthers P and Shin C C 1985 *Phys. Lett.* **165B** 209
[14] Shin C C 1986 *Phys. Rev.* D **33** 339
[15] Harris T E 1963 *The Theory of Branching Processes* (Berlin: Springer)
Athreya K B and Ney P E 1972 *Branching Processes* (Berlin: Springer)
[16] Kobayashi T 1988 *J. Phys. A: Math. Gen.* **21** 3723